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A 2-SPHERE IN E^3 WITH VERTICALLY CONNECTED INTERIOR IS TAME

BY

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ABSTRACT. A set X in E^3 is said to have *vertical number n* if the intersection of each vertical line with X contains at most n components. The set X is said to have *vertical order n* if each vertical line intersects X in at most n points. A set with vertical number 1 is said to be *vertically connected*. We prove that a 2-sphere in E^3 with vertically connected interior is tame. This result implies as corollaries several previously known taming theorems involving vertical order and vertical number along with several more general and previously unknown results.

1. Introduction. The theorem of the title has apparently been proved independently by Cobb. The Fox-Artin wild sphere [12, Example 1.2] (cf. [7, §3.3]) can be described in E^3; so that its interior has vertical number 2; thus the theorem cannot be generalized in this direction. However we shall show in §2 that a 2-sphere whose interior has vertical number 2 is tame from its interior (Theorem 2.6). That the number 2 is the best possible for this conclusion can be seen using a slight modification of the Fox-Artin construction (which runs the knotted feeler inward rather than outward). Questions of the type discussed in this paper are mentioned in [7, §9]; and previous related results appear in [9], [10], [15], [16], and [17].

Notation.

$$\mathrm{Cl}(X) = \text{closure of } X,$$

$$E^3 = E^2 \times E^1, \quad \pi\colon E^3 \to E^2, \quad \pi'\colon E^3 \to E^1,$$

$$B^2 = \{x \in E^2 \mid |x| \leq 1\}, \qquad S^1 = \mathrm{Bd}\, B^2 = \{x \in E^2 \mid |x| = 1\}.$$

Note. $B^2 = [0, 1] \cdot S^1 = \{r \cdot s \mid r \in [0, 1], s \in S^1\}$ ($r \cdot s$ denotes the scalar multiplication of r times the vector s).

Definition. A subset X of E^3 is called vertically connected if, for each $x \in X$, $(X \cap \pi^{-1}\pi(x))$ is connected.

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Lifting Lemma. *Suppose that U is a vertically connected open set in E^3 and that G is a finite graph in the open subset πU of E^2. Then there is a finite graph $G' \subset U$ such that $\pi | G'$ takes G' homeomorphically onto G. Furthermore, if $x_1, x_2, \cdots, x_n$ are finitely many points of G and $x_1', x_2', \cdots, x_n'$ are points of U such that $\pi x_i' = x_i$ $(i = 1, \cdots, n)$, then G' may be chosen to contain the set $\{x_1', x_2', \cdots, x_n'\}$.*

Proof. Exercise.

Main theorem. *If S is a 2-sphere in E^3 with vertically connected interior U, then S is tame.*

Proof. From the standpoint of current theory, the difficulty in the main theorem arises from the fact that a vertical line which intersects S but not U may intersect S in a set which is not connected. The proof has therefore, two main parts: the easier Part 1 which shows that S is tame if $\mathrm{Bd}(\pi U)$ is a simple closed curve (a case in which the previously mentioned difficulty does not arise; cf. the proof of 1.1) and a harder Part 2 in which it is shown that there is a 2-sphere S' in E^3 with vertically connected interior U' such that S and S' are equivalently embedded in E^3 and such that $\mathrm{Bd}(\pi U')$ is a simple closed curve. The two parts will be preceded by an initial Part 0 in which the general properties of πU and $\mathrm{Bd}(\pi U)$ will be established.

Part 0. *Properties of πU and* $\mathrm{Bd}(\pi U)$.

0.1. The set πU is a bounded, connected, and simply connected open subset of E^2.

Indeed, πU is bounded, connected, and open since the projection π preserves the corresponding properties of U.

We show that πU is simply connected by showing that each simple closed curve J in πU is nullhomologous in πU. By the Lifting Lemma, there is a simple closed curve J' in U such that π takes J' homeomorphically onto J. Since U has trivial first homology [21], J' is the boundary of a singular 2-chain M in U. Thus J bounds the singular 2-chain πM in πU; i.e., J is nullhomologous in πU.

0.2. The set $\mathrm{Bd}(\pi U)$ is a locally connected (compact metric) continuum.

The set $\mathrm{Bd}(\pi U)$ is a continuum since πU is a bounded, connected, and simply connected open subset of E^2. Also, $\mathrm{Bd}(\pi U)$ is locally connected; for otherwise there are an $\epsilon > 0$ and a sequence $\{p_i\}_{i=1}^{\infty}$ of points in πU converging to a point $p \in \mathrm{Bd}(\pi U)$ such that no two of the points of the sequence can be joined by an ϵ-arc in πU. For each i, let q_i be a point of U such that $\pi q_i = p_i$. Passing to a subsequence, we may assume that the sequence $\{q_i\}$ converges. Since U is 0-ulc [21, p. 66], it follows that each pair of points q_i and q_j with sufficiently large subscripts can be joined by an ϵ-arc in U. The projection under π of such an ϵ-arc contains an ϵ-arc in πU which joins p_i and p_j. This contradiction proves that $\mathrm{Bd}(\pi U)$ is locally connected.

0.3. If $x \in \pi U$, then $S \cap \pi^{-1}(x)$ has precisely two components.

Clearly $S \cap \pi^{-1}(x)$ has at least two components, an uppermost component C_1 and a lowermost component C_2. Pick $x' \in U$ such that $\pi x' = x$. It suffices to show that the minimal vertical line segments X_1 and X_2 joining x' to C_1 and C_2, respectively, lie in $S \cup U$. Since C_i $(i = 1, 2)$ is in Cl U, there is a sequence $\{x_j^i\}_{j=1}^{\infty}$ of points in U converging to any given point $x_i \in C_i$. Let P denote the horizontal plane $E^2 \times \{\pi'(x')\}$ through x'. For j sufficiently large, the vertical line through x_i^j intersects P in U since U is open. Since U is vertically connected, the vertical segment X_j^i joining x_j^i to P lies in U for j large. But $X_i \subset \lim_{j\to\infty} X_j^i$. Hence $X_i \subset S \cup U$ as desired.

Part 1. *Special case*. If $\mathrm{Bd}(\pi U)$ is a simple closed curve, then S is tame. (The special hypothesis that $\mathrm{Bd}(\pi U)$ is a simple closed curve is to be assumed in Steps 1.1–1.4.)

1.1. The set $X = S \cap \pi^{-1}\mathrm{Bd}(\pi U)$ is a tame nondegenerate continuum, hence is a *-taming set.

The set X is tame since it lies in the cylinder $\pi^{-1}\mathrm{Bd}(\pi U)$. That X is connected will follow from general properties of monotone mappings $(\pi | X)$ once we show that, for each $x \in \mathrm{Bd}(\pi U)$, $S \cap \pi^{-1}(x)$ is connected. Let $p, q, \in S \cap \pi^{-1}(x)$. It suffices to show that the vertical segment $[p, q]$ joining p to q lies in Cl U. Pick sequences $\{p_i\}$ and $\{q_i\}$ in U converging to p and q respectively. Since $\mathrm{Bd}(\pi U)$ is a simple closed curve and $\pi p = \pi q$, there is a null sequence $\{A_i\}$ of arcs in πU such that A_i joins πp_i to πq_i. By the Lifting Lemma, there is, for each i, an arc A_i' in U such that A_i' joins p_i to q_i and $\pi A_i' = A_i$. It follows that $[p, q] \subset \lim_{i\to\infty} A_i' \subset S \cup U$ as desired.

1.2. Remark. By 0.3 and 1.1, $S \cup U$ is vertically connected; hence S is tame by [17]. However, the proof that S is tame in our situation is short and we include it (1.3 and 1.4).

1.3. The set $E^3 - U$ is a *-taming set; hence S is tame from U.

Let X_i be the closed subset of $E^3 - U$ that is formed by taking the union of all closed vertical segments in $E^3 - U$ of length at least $1/i$. The set X_i is a *-taming set by [9, Theorem 5]. It is an immediate consequence of 0.3 that $E^3 - U = \bigcup_{i=1}^{\infty} X_i$. Hence $E^3 - U$ is a *-taming set by [8, Theorem 3.7]. It follows from the definition of *-taming set [8, p. 429] that S is tame from U.

1.4. The set $S \cup U$ is a *-taming set; hence S is tame from Ext S.

Let Y_i be the union of all closed vertical segments in $S \cup U$ of length at least $1/i$. It is an immediate consequence of 0.3 and 1.1 that $S \cup U$ is the union of the *-taming sets X, Y_1, $Y_2, \cdots$. Thus $S \cup U$ is a *-taming set by [8, Theorem 3.7], and S is tame from Ext S.

Part 2. *Reduction to the special case.* There is a 2-sphere S' in E^3 with vertically connected interior U' such that S and S' are equivalently embedded in E^3 and such that $\mathrm{Bd}(\pi U')$ is a simple closed curve.

The idea in Part 2 is as follows: Since $\mathrm{Bd}(\pi U)$ is locally connected, it has a certain "nice" singular collar in πU (cf. 2.1). This collar may be used to describe a horizontal push (cf. 2.2) of $S \cup U$ that moves $S \cup U$ from $\mathrm{Cl}(\pi U) \times E^1$ into $\pi U \times E^1$. This horizontal push moves $S \cap \pi^{-1}\mathrm{Bd}(\pi U)$ so that its projection under π is a simple closed curve. The image S' of S under this push is the desired 2-sphere.

2.1. *Singular collar theorem.* There is a continuous function $f\colon B^2 \to \mathrm{Cl}(\pi U)$ such that $f \mid \mathrm{Int}\, B^2$ takes $\mathrm{Int}\, B^2$ homeomorphically onto πU and such that $f^{-1}(x)$ is a totally disconnected subset of $S^1 = \mathrm{Bd}\, B^2$ for each $x \in \mathrm{Bd}(\pi U)$. (Since $f|\mathrm{Int}\, B^2$ takes $\mathrm{Int}\, B^2$ homeomorphically onto πU, it follows automatically from invariance of domain that $f(S^1) = \mathrm{Bd}(\pi U)$.)

Except for the requirement that $f^{-1}(x)$ be totally disconnected this is just [18, p. 186]. Let $f'\colon B^2 \to \mathrm{Cl}(\pi U)$ be a map satisfying all requirements except the disconnectivity property. Let $(\sim)$ be the upper semicontinuous decomposition of B^2 whose only nondegenerate elements are the nondegenerate components of the sets $(f')^{-1}(x)$ $(x \in \mathrm{Bd}(\pi U))$. There is a commutative diagram

$$\begin{tikzcd} B^2 \arrow[rr, "f'"] \arrow[dr, "p"'] & & \mathrm{Cl}(\pi U) \\ & B^2/(\sim) \arrow[ur, "f"'] & \end{tikzcd}$$

of continuous maps, where p is the projection map and where the map f is uniquely determined by p, f', and commutativity. Since $B^2/(\sim)$ is homeomorphic with B^2, we may identify $B^2/(\sim)$ with B^2 and use f as the map required by the theorem. (Alternatively, we may prove 2.1 by an appeal to Carathéodory's theory of prime ends.)

2.2. *Construction of a horizontal push H_* of $S \cup U$ from $S \cup U$ into $\pi U \times E^1$.* Choose a positive integer N so large that $S \cup U \subset E^2 \times [-N, N]$. Define the following continuous functions and sets:

$F\colon B^2 \times E^1 \to \mathrm{Cl}(\pi U) \times E^1$, the continuous function defined by the formula $F(x, t) = (f(x), t)$ $(x \in B^2, t \in E^1)$.

X, the closed set $[\mathrm{Cl}(F^{-1}(U))] \cup [\tfrac{1}{2} \cdot B^2 \times E^1]$.

$g\colon S^1 \times E^1 \to [0, 1]$, the continuous function defined by the formula $g(s, t) = 1 - [\tfrac{1}{2} \cdot \rho((s, t), X)]$ $(s \in S^1, t \in E^1)$.

C, the 3-cell $B^2 \times [-N, N]$.

$G: C \to E^3$, an embedding defined by radially shrinking C according to the formula $G(r \cdot s, t) = (r \cdot g(s, t) \cdot s, t)$ $(r \in [0, 1], s \in S^1, t \in [-N, N])$.

C_1, the 3-cell $G(C)$.

$H: C_1 \to E^3$, an embedding defined by radially shrinking C_1 according to the formula $H(x, t) = (\frac{1}{2} \cdot x, t)$ $((x, t) \in C_1 \subset B^2 x E^1)$.

C_2, the 3-cell $H(C_1)$.

Note that H is a horizontal push which takes the 3-cell C_1 onto the 3-cell C_2 and preserves vertical lines. Note further that $F(C_1) \supset U$, hence that $F(C_1) \supset \mathrm{Cl}\ U$, and that $F(C_2) \subset \pi U \times E^1$. We shall prove in 2.3 that $F|C_1$ is 1-1 hence that $H_* = (FHF^{-1}|FC_1)$: $F(C_1) \to F(C_2)$ defines a homeomorphism and a horizontal push which takes the 3-cell $F(C_1)$ onto the 3-cell $F(C_2)$. The map $H_*|\mathrm{Cl}\ U$ is the desired horizontal push from $\mathrm{Cl}\ U$ into $\pi U \times E^1$.

2.3. The map $F|C_1$ is 1-1.

Suppose not. Then it follows from the construction of F and C_1 that any two points of C_1 identified by F are of the form (s_1, t) and (s_2, t), where each is in $\mathrm{Cl}(F^{-1}(U))$ and where s_1 and s_2 are in S^1. Pick sequences $\{x_i\}$ and $\{y_i\}$ in $F^{-1}(U)$ converging to (s_1, t) and (s_2, t), respectively. The sequences $\{Fx_i\}$ and $\{Fy_i\}$ lie in U and converge to the point $F(s_1, t) = F(s_2, t)$. Since U is 0-ulc, there is a null sequence $\{A_i\}$ of arcs in U such that A_i joins Fx_i to Fy_i. The sequence $\{\pi A_i\}$ of paths in πU converges to $\pi F(s_1, t)$. Hence, by the continuity of $f: B^2 \to \mathrm{Cl}(\pi U)$,

$$\limsup_{i \to \infty} \{\pi F^{-1} A_i\} \subset f^{-1} \pi F(s_1, t).$$

But $\limsup_{i \to \infty} \{\pi F^{-1} A_i\}$ contains at least one of the arcs in S^1 from s_1 to s_2. This contradicts the fact that $f^{-1}\pi F(s_1, t)$ is totally disconnected and proves that $F|C_1$ is 1-1.

2.4. The homeomorphism $H_* = (FHF^{-1}|FC_1)$: $FC_1 \to FC_2$ of 2.2 can be extended to a homeomorphism $H_*: E^3 \to E^3$.

Indeed, $H_*|FC_1: FC_1 \to FC_2$ is a homeomorphism between 3-cells. These 3-cells are tame from their exteriors since each horizontal slice in each of these 3-cells is a disk (cf. [12], [14]). Thus the extension to all of E^3 is immediate.

2.5. The 2-sphere $S' = H_*(S)$ satisfies the requirements of Part 2.

We leave the checking of 2.5 to the reader. We comment only that $\mathrm{Bd}(\pi U') = \mathrm{Bd}(\pi\ \mathrm{Int}\ S')$ is the simple closed curve $f(\frac{1}{2} \cdot S^1)$.

Result 2.5 completes the reduction to the special case of Part 1. The proof of the main theorem is complete.

2. **Applications.** Our main theorem serves as the basis for proofs of a number of related theorems. An immediate consequence is the following generalization of the main result in [15].

Theorem 2.1. *If a 2-sphere S in E^3 has vertical number 3, then S is tame.*

Proof. We shall reduce this theorem to the main theorem by showing that Int S is vertically connected. Suppose that there are two components U and V in the intersection of a vertical line L with Int S. Let $\{x_i\}$ be a sequence of points in Ext S converging to a point x in L such that x is between U and V in L. Now a vertical line through x_i, for i sufficiently large, will have at least four components in its intersection with S. This contradicts the fact that S has vertical number 3.

Remark. The proof of Theorem 2.1 shows that S is tame with the weaker hypothesis that it have vertical number 3 only relative to those vertical lines that intersect Int S.

We next focus our attention on a disk in E^3 having vertical number 3. Not all such disks are tame (see [13, Example 1.2] again); however, it follows from Theorem 2.3 that such a disk can be wild only at its boundary. We need the following technical lemma to establish Theorem 2.3.

(2.2) **Localization lemma.** *Suppose that D is a disk lying on a 2-sphere S in E^3, and that U is a component of $E^3 - S$ with the following property:*

> *If p is a point of* Int D *that lies in no vertical (nondegenerate) interval in* $\mathrm{Cl}(U)$, *then p lies in the interior of a tame disk in D.*

Then S is locally tame from $E^3 - \mathrm{Cl}(U)$ at each point of Int D.

Proof of (2.2). Let $p \in$ Int D. We may assume that Bd D is tame since a smaller disk in D with p in its interior exists with this property [5]. Let X_i be the union of all closed vertical intervals in $\mathrm{Cl}(U)$ that have a diameter no less than $1/i$ and that intersect D. It is an exercise to see that X_i is closed, and it follows from [9, Theorem 5] that X_i is a $*$-taming set, for each i. If p is a point of Int D that does not lie in $\bigcup_{i=1}^{\infty} X_i$, then by the hypothesis on U, it follows that p lies in the interior of a tame disk in D. A countable collection $\{D_1, D_2, D_3, \cdots\}$ of such tame disks suffices to cover the set of all such points. Now we let

$$X = \left(\bigcup_{i=1}^{\infty} X_i\right) \cup \left(\bigcup_{i=1}^{\infty} D_i\right) \cup (\mathrm{Bd}\ D),$$

and we claim that X is closed. To see this, suppose that q is a limit point of X that is not in X, and note that q is not in D. Thus q is a limit point of $(\bigcup_{i=1}^{\infty} X_i)$, and there must exist a null sequence $\{F_i\}$ of vertical intervals, each intersecting D, such that q belongs to $\lim_{i\to\infty} F_i$. But then q would be a limit point of D.

Since X is closed and is the countable union of *-taming sets [8, Theorem 3.7], we see that X is also a *-taming set [8, Theorem 3.7]. We approximate $S - X$ with a locally polyhedral surface [3, Theorem 1] to obtain a 2-sphere S' that contains D and such that $S' \cap X = S \cap X$, and it follows from the definition of *-taming sets that S' is tame from the complementary domain of S' that does not intersect X. Thus S is locally tame from $E^3 - \mathrm{Cl}(U)$ at p, and hence S is locally tame from $E^3 - \mathrm{Cl}(U)$ at each point of Int D.

Theorem 2.3. *If a disk F in E^3 has a tame boundary and has vertical number 3, then F is tame.*

Proof. We shall show that F is locally tame at each of its interior points, and the theorem will then follow from [11] (cf. [8, Theorem 3.7]) since $\mathrm{Bd}\, F$ is tame. Let $p \in \mathrm{Int}\, F$, and choose a subdisk D of F and a 2-sphere S such that $p \in \mathrm{Int}\, D \subset S$, $\mathrm{Bd}\, D$ is tame, and S is locally tame at the points of $S - D$ [4, Theorem 5]. Let $U = \mathrm{Ext}\, S$ and note that to show D locally tame from the side facing Int S, it suffices, from (2.2), to show that if q is a point of $\mathrm{Int}\, D$ that lies in no vertical interval in $\mathrm{Cl}(U)$, then q lies in a tame disk in D. Let q be such a point. We may assume that D is small enough that the intersection of D with the vertical line L through q is $\{q\}$, for such a smaller disk is easily found on S. Let M be a 2-sphere in the shape of a right circular cylinder with a horizontal top and bottom such that $q \in \mathrm{Int}\, M$, $M \cap S \subset \mathrm{Int}\, D$, and $D \cap M$ lies in the vertical lateral side of M. Let H be the collection of vertical intervals in $M \cap D$, and let G be the upper semicontinuous decomposition of E^3 whose nondegenerate elements are the intervals in H. Bing [2, Theorem 5] has not only proven that the decomposition space E^3/G is homeomorphic to E^3 but that the projection map $f\colon E^3 \to E^3/G$ may be taken to preserve vertical lines. By [19] $S' = f(S)$ is a 2-sphere and $f(D) = D'$ is a disk. Let $C = S' \cup f(U)$, and let K be the boundary of a component of $f(M) \cap \mathrm{Int}\, C$. From [16, p. 676–677] it follows that K is a simple closed curve in $f(M)$. Since q lies in no vertical interval in C, L cannot pierce S' at q. From this and the fact that $L \cap D = \{q\}$ we know that K cannot link L. Thus K bounds a vertical disk in $f(M)$. Now $f(M) \cap D'$ has vertical order 3 (D has vertical number 3), and no two of these vertical disks have intersecting interiors. Thus the collection $\{E_1, E_2, E_3, \cdots\}$ of such disks forms a null sequence, for otherwise some subsequence would converge to a (nonexistent) vertical interval in $f(M) \cap D'$. Now a 2-sphere T having vertical number 3 can be formed using the component N of $D' - f(M)$ containing q and filling in its holes either with an appropriate disk from the collection $\{E_i\}$ or with a nonseparating component of $D' \cap f(M)$. From Theorem 2.1, T is tame. Since f is a homeomorphism near q, it follows that D is locally tame at q. Now it follows from (2.2) that $\mathrm{Int}\, D$ is locally tame from $\mathrm{Int}\, S$, so that F is locally tame at p from one side.

Since S is locally polyhedral away from D it is clear that another 2-sphere Q can be constructed so that $D \subset Q$, Q is locally tame modulo D, and Q is locally tame at p from $\operatorname{Ext} Q$. If Q is now allowed to play the role of S in the above proof, we see that Q is also locally tame at p from its interior. Thus F is locally tame at p from both sides and hence locally tame at each of its interior points.

Remark. Theorems 2.4 and 2.5 generalize the main results in [15] and [16], respectively. All 2-manifolds considered here are to be connected and without boundary.

Theorem 2.4. *If a compact 2-manifold M in E^3 has vertical number 3, then M is tame.*

Proof. Each disk in M has vertical number 3, so Theorem 2.4 follows from Theorem 2.3 and [1], [20].

Theorem 2.5. *If M is a compact 2-manifold in E^3 having vertical number 5, and if V is the bounded component of $E^3 - M$, then M is tame from V.*

Proof. Let $p \in M$. There is a disk D in M with p in its interior and there is a 2-sphere S containing D such that D lies in the boundary of $(\operatorname{Int} S) \cap (E^3 - \operatorname{Cl}(V))$ [6, Theorem 1]. Now it will follow from (2.2) that S is locally tame from $\operatorname{Ext} S$ at p once we show that each point q of $\operatorname{Int} D$ either lies in a tame disk in D or lies in a vertical interval in $S \cap \operatorname{Int} S$. Of course M is then locally tame from V at p.

Suppose q is a point of $\operatorname{Int} D$ that does not lie in a vertical interval in $S \cup \operatorname{Int} S$. Then there exist two disjoint vertical open intervals I and I', with q as their common endpoint, both lying in V. Let B and B' be open balls centered at points of I and I', respectively, such that $B \cup B' \subset V$. Choose a disk D' in D such that $q \in \operatorname{Int} D'$, $\operatorname{Bd} D'$ is tame, and each vertical line intersecting D' also intersects both B and B'. Now it follows from the fact that M has vertical order 5 that D' has vertical order 3. Then D' is tame (Theorem 2.3). Now (2.2) applies and the result follows.

Theorem 2.6. *If M is a compact 2-manifold in E^3 and V is a component of $E^3 - M$ with vertical number 2, then M is tame from V.*

Proof. Let $p \in M$, and let D be a disk in M that lies on a 2-sphere S in E^3 [4, Theorem 5]. From [6, Theorem 1] we see that S and D can be chosen so that D lies on the boundary of $V \cap \operatorname{Int} S$. We shall use (2.2) to show that S is locally tame at each point of $\operatorname{Int} D$ from $U = \operatorname{Int} S$. This would make M locally tame at p from V and the result would then follow from [1] or [20].

To apply (2.2) we need to show that if a point q of $\operatorname{Int} D$ lies in no vertical interval in $E^3 - U$, then q lies in the interior of a tame disk in D. Let N be an open ball centered at q such that $N \cap (S \cup M) \subset \operatorname{Int} D$. Let L be the vertical line through q, and notice that q is the common endpoint of two disjoint open intervals I and I' in $V \cap N \subset U$. Let B and B' be two open balls centered at points of I and I', respectively, such that $B \cup B' \subset N \cap V$. Choose a disk D' in D, with tame boundary, such that $q \in \operatorname{Int} D'$ and every vertical line intersecting D' also intersects both B and B'. We claim that D' has vertical number 2. When we establish this, it will follow from Theorem 2.3 that D' is tame and the result will follow.

Suppose there is a vertical line L' intersecting D' such that $L' \cap D'$ has three components C_1, C_2, C_3. There must be vertical intervals I_1 and I_2 in $B \cap L'$ and $B' \cap L'$, respectively. We may assume the labeling so that the sets are ordered from top to bottom on L' as follows: $I_1 < C_1 < C_2 < C_3 < I_2$. Neither of the two open intervals on L' between C_1, C_2 and between C_2, C_3 can intersect V because V has vertical number 2. Thus there must be two disjoint open balls K and K' with centers on L' between C_1, C_2 and C_2, C_3 in E^3, respectively, such that $K \cup K' \subset E^3 - \operatorname{Cl} V$. Choose a point f in $\operatorname{Int} S$ so close to C_2 that the vertical line L'' through f intersects each of B, K, K', B'. Then $L'' \cap V$ will have 3 components. Of course this is a contradiction.

Theorem 2.7. *If N is a compact 2-manifold in E^3 such that the bounded component V of $E^3 - N$ has vertical number* 1, *then N is tame.*

Proof. It follows from Theorem 2.6 that N is tame from V since V has vertical number 2. The set $W = E^3 - \operatorname{Cl}(V)$ might not have vertical number 2, so we are forced to go a different route to show that N is locally tame from W.

Let $p \in N$. There exist a disk D and a 2-sphere S such that $p \in \operatorname{Int} D \subset S$, $\operatorname{Bd} D$ is tame, and D lies in the boundary of $V \cap \operatorname{Int} S$ [6, Theorem 1]. Again we depend on (2.2) to see that S is locally tame from $\operatorname{Ext} S$ at p. To show that (2.2) applies we must prove that if a point q of $\operatorname{Int} D$ does not lie in a vertical interval in $S \cup \operatorname{Int} S$, then q lies in a tame disk in D. If q is such a point, then we select a 2-sphere M having the shape of a right circular cylinder with a horizontal top and bottom not intersecting N such that $q \in \operatorname{Int} M$ and $M \cap (N \cup S) \subset \operatorname{Int} D$. As in the proof of Theorem 2.3, we apply [2, Theorem 5] to shrink the vertical intervals in $M \cap S$ to obtain a 2-sphere $S' = f(S)$, a disk $D' = f(D)$, and a 2-sphere $M' = f(M)$. Let $C = S' \cup f(\operatorname{Int} S)$, and consider the boundary K of a component X of $M' \cap \operatorname{Int} C$. It follows from the techniques given in [16, pp. 676–677] that K is a simple closed curve. To see that K bounds a vertical disk D_k on M it suffices to show that K does not link the vertical line L through q. If K links L, then L must intersect $V \cap \operatorname{Int} S'$ near the subdisk E of D' bounded by K. But $L \cap V$ is connected and does not have q as a limit point, since V has vertical number 1 and q does not lie in a vertical interval in $L \cap (S' \cup \operatorname{Int} S')$. Thus a vertical line L'

sufficiently close to L and containing a point e of V near q would have two components in its intersection with V. Hence we have a contradiction to the fact that V has vertical number 1. Thus K does not link L and K bounds a vertical disk in $M' \cap \operatorname{Int} C$.

Since $M' \cap \operatorname{Int} C$ has at most a countable number of such components X, there is a countable collection $\{D_1, D_2, D_3, \cdots\}$ of such vertical spanning disks D_k for C. Since $\bigcup_{i=1}^{\infty} \operatorname{Int} D_i$ has vertical number 1 (because $\operatorname{Int} D_i \subset V$) and since $M' \cap D'$ contains no vertical interval, it follows that $\{\operatorname{Int} D_i\}$ is a null sequence of pairwise disjoint open disks such that $D_i \cap S = \operatorname{Bd} D_i$, for each i. Let Q be the component of $(\operatorname{Int} C) \cap (\operatorname{Int} M')$ having q on its boundary, and note that since $Q \subset V$, Q has vertical number 1. The boundary T of Q is a 2-sphere whose interior is Q. Thus T is tame by the main theorem, and it follows that q lies in a tame disk in D. Theorem 2.7 then follows from (2.2) and [1] or [20].

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